Secondary 3 Additional Mathematics Performance Task (Individual Component)

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Recently, there have been many flash floods in SST, some of them happening near the vicinity of SST itself. The Ulu Pandan Canal is one of the canals near SST, and we are trying to find methods to reduce the risk of flash floods there. To do this, we will use mathematical modeling to help find a solution to minimise the problem.

1.1. Mathematical Model

First, we need to create a mathematical model to convert this problem to mathematical functions and variables that we can manipulate. We can use coordinate geometry to make a diagram of the cross-section of the canal and the graphing calculator Desmos to draw it out. Before making our model, we need to make some assumptions on the model itself. First, we can assume that the canal is a perfect trapezium prism, and the non-parallel sides are equal. (Fig. 1.1)

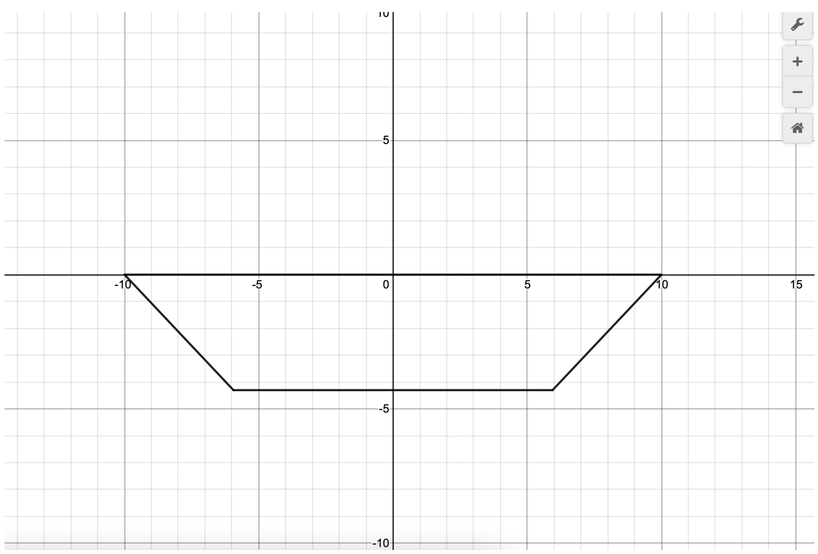

Fig. 1.1 - A diagram of the cross-section of the canal, which is assumed to be a trapezium where the non-parallel sides are equal (also known as an isosceles trapezium)

We also assume the floor around the canal is 0 on the *y*-axis, with the canal being inset downwards.

Next, we need to find what variables we need to find for the trapezium itself. We can use a variable for the depth of the canal [*d*], another one for the width of the canal [*w*], and another for the slope of the non-parallel sides of the trapezium [*m*]. (Fig. 1.2)

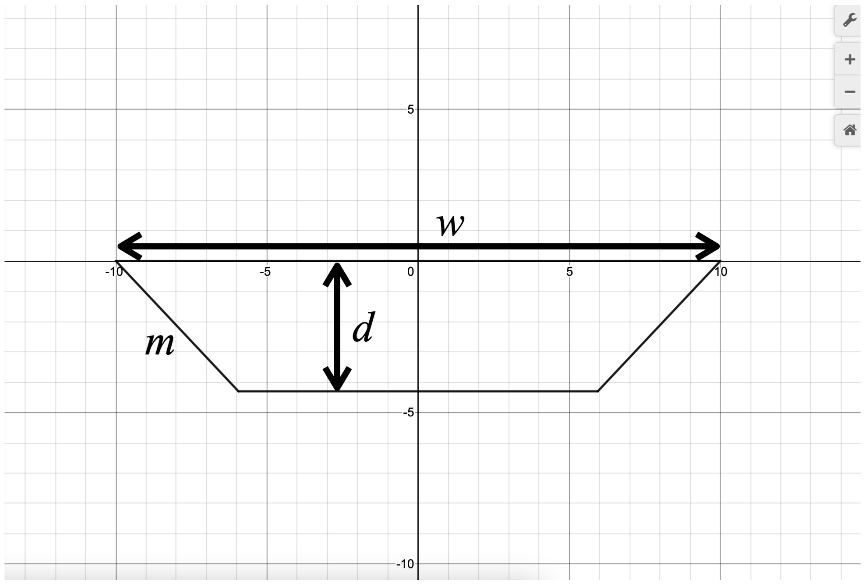

Fig. 1.2 - The trapezium labeled with the variables stated earlier

Thus, with the assumptions and variables listed, we can now use functions to create the trapezium in Desmos.

1.2. Forming the Model

The trapezium can be split up into two parts; the two side slanted lines and the bottom line of the canal. The top line of the canal can be omitted as there will not be any material there.

For the bottom line of the canal, we can simply make a horizontal line where it cuts the *y* axis at *-d*.

Thus, our function is just

$$x = \quad - d$$

For the two slanted lines, we can make use of an absolute function, since when graphed out, it gives us two lines, the left side of the graph having a negative slope, while the right side has a positive slope, which is what we want for the lines in the trapezium.

Thus, we can start with a simple absolute function with *x,*

$$y = |x|$$

We can then add the variable *m* to our function since when we take the absolute value of mx, we change the slope of each side of the graph.

$$y = |mx|$$

Currently, this graph only intersects the *x*-axis at one point, but we want to shift it down such that the distance between the two points of intersection between the function and the *x*-axis is w.

When we want the distance between the points of intersection to be *w*, we want the points of intersection to be (0.5w, 0) and (-0.5w, 0). For now, let us try to remove the absolute function and focus only on what is inside the function itself.

$$y = mx$$

As shown, the function inside the absolute function is a linear one. Since we want it to intersect at the *x*-axis at a certain point, we just want to find the *x*-intercept of the line.

Thus, we can substitute in 0.5w and add or subtract from that to let y = 0.

$$y = m(0.5w)$$
$$y = 0.5mw$$

Thus, to make y = 0, we can minus off the same value to give us

$$y = 0.5mw - 0.5mw$$

We now know that we must subtract 0.5mw from the absolute function to lower it to the right level.

$$y = |mx| - 0.5mw$$

Now with our two functions, if we were to plot them out, we get the following (Fig. 2.1)

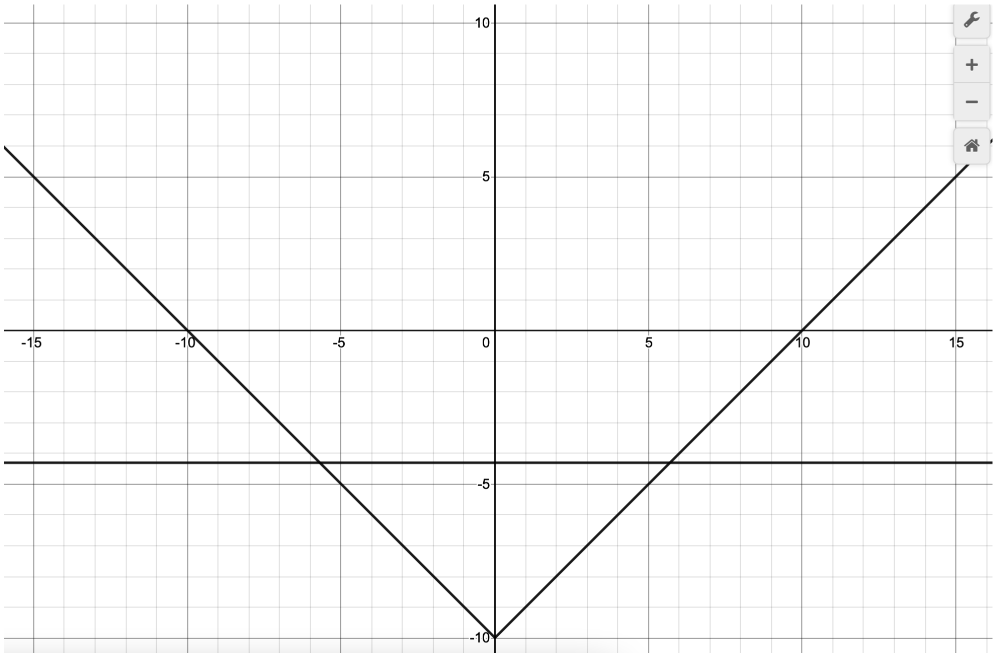

Fig. 1.3 - The graphs plotted out in Desmos

Our trapezium is starting to take shape, but we can use inequalities to cut the lines at specific points to form the trapezium.

We simply want the absolute function to be included in the part where it is part of the canal depth. Thus, the function is now

$$y = |mx| - 0.5mw \{0 > y >- d\}$$

For the horizontal line, we would want the inequality to be where it intersects with the absolute function itself, which is

$$|mx| - 0.5mw =- d$$
$$|mx| - 0.5mw + d = 0$$

We then need to remove the absolute function again so that we can use it on two sides of an inequality, which leaves us with two equations

$$mx - 0.5mw + d = 0 \text{ and} - mx - 0.5mw + d = 0$$

We also want to factorise out the x value so that we only get a number, and if we focus on one of the equations, we get

$$mx - 0.5mw + d = 0$$
$$- 0.5mw + d =- mx$$
$$\frac{-0.5mw+d}{-m} = \frac{-mx}{-m}$$
$$x = \frac{-0.5mw+d}{-m}$$

For the other equation, we can change the negative sign of the variable *m* in the denominator to get the other value of *x*.

Now, we have the two values of *x*, being

$$x = \frac{-0.5mw+d}{-m},\ x = \frac{-0.5mw+d}{m}$$

We can now put these values into our function inequality, giving us

$$y =- d\{\frac{-0.5mw+d}{-m} > x > \frac{-0.5mw+d}{m}\}$$

Plotting the two graphs, we get our final trapezium (Fig. 2.2)

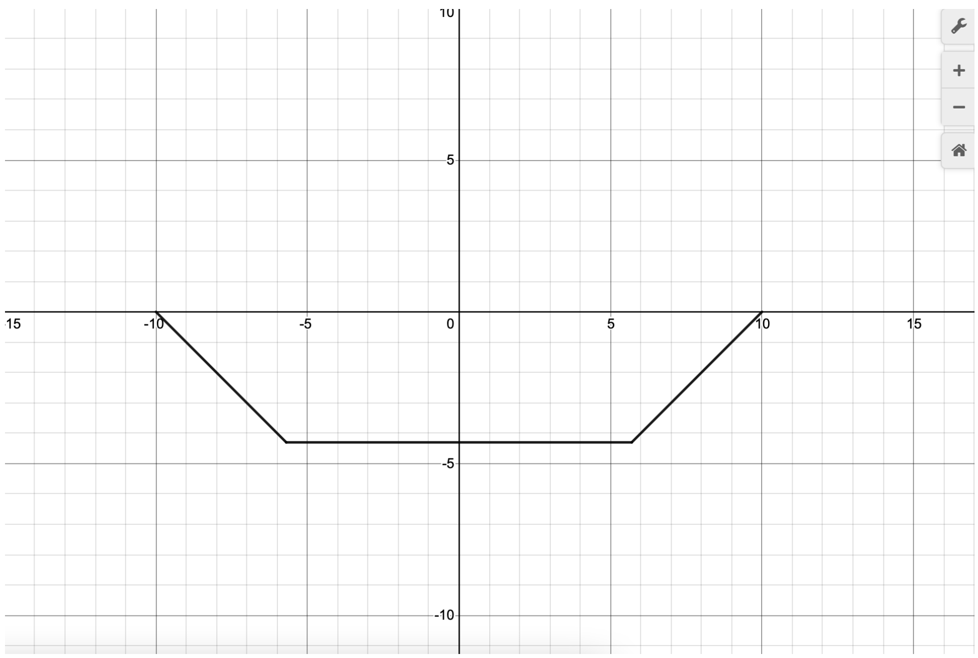

Fig. 1.4 - The graphs plotted out in Desmos with the correct inequalities

We can now change the variables to see our canal shape change in real-time, visualising other cross-sections.

Now, we need to substitute the correct values into the graph to create a scale model of the canal cross-section.

1.3 Gathering Data

The data that we need to get for our model to be to scale is the width [w], the depth [d] and the gradient [m] of the sides of the canal.

The depth is the easiest one to substitute since we are already given that the canal's depth is 5m.

Next, we need to find the width of the canal. We can use Google Map's scale reference to measure the canal's width. Using a screenshot of Google Maps(Fig. 1.5) and the inbuilt Preview program, we can count the number of pixels to get the width.

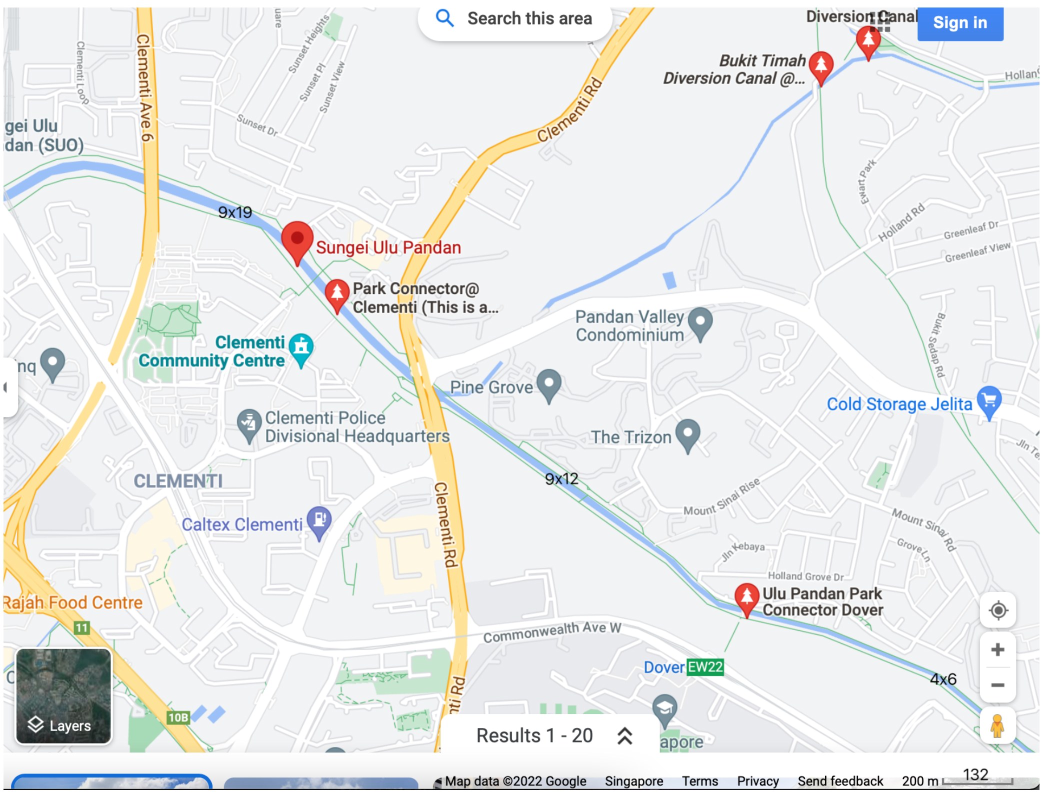

Fig. 1.5 - Reference screenshot of Google Maps, with labeled pixel distances

Whenever the selection box is drawn, Preview gives the pixel length and width of the selection box itself. We can quickly find the pixel length without having to count them manually.

Thus, if we select over the reference scale in the bottom right corner of the screenshot, Preview gives us a pixel count of 132. Therefore, 200 meters in the real world is 132 pixels on our map. We can then measure the canal's width by drawing a selection box where the diagonal of the rectangle is somewhat perpendicular to the canal itself. We then repeat this three more times at different points of the canal where they are thicker or thinner.

As shown in Fig. 1.5, three points on the canal were picked out, and their selection boxes were labeled there.

Now, using this information, we can estimate the canal's width. We will use the selection box of 9 by 12 as an example.

Since we want to find the diagonal of the selection box, we are just finding the hypotenuse of a right-angled triangle of base 9 and height 12. To find the hypotenuse, we can use Pythagorean's Theorem.

$$a^2 + b^2 = c^2$$
$$9^2 + 12^2 = c^2$$
$$c = \sqrt{9^2 + 12^2}$$

If we repeat this for the other two points and take the mean value of them to get an estimate of the canal's width, we get

$$t = \frac{\sqrt{9^2+12^2}+\sqrt{9^2+19^2}+\sqrt{4^2+6^2}}{3}$$

(t is a temporary variable to make the calculations look cleaner)

The variable *t* now gives us an accurate reading of the width, but it has not been converted to meters yet. Using the 200m to 132 px (pixels) ratio, we can find the canal's width in meters.

$$200m: 132px$$
$$200/(132/t)m: 132/(132/t)\ px$$
$$\frac{200}{\frac{132}{t}}m: t\ px$$

Thus, the canal's width is $\frac{200}{\frac{132}{t}}m$, which is

$$\frac{200}{\frac{132}{\frac{\sqrt{9^2+12^2}+\sqrt{9^2+19^2}+\sqrt{4^2+6^2}}{3}}}m = 21.8m\ (3.s.f)$$

We can then substitute this into our variable *w* for the canal's width.

Lastly, we need to find out the variable [*m*] for the gradient of the sides of the canal. This is much more difficult to get since the gradient needs the rise and run of the line itself. However, there is a possible method to find the data.

Since we have an image of the canal, we can use camera matching software to recreate the image scene in 3D and measure the 3D models to find the gradient. The software for this is called fSpy, which, although being mainly used for 3D art and design, is quite helpful in this scenario. fSpy lets the user mark out lines that are along the *x*, *y* or *z*-axis, and it calculates the vanishing points of the scene to work out the camera's location in 3D space, its rotation, and its focal length. The camera data can then be imported into a 3D software, like Blender, where we can model from the reference image itself.

Thus, we will use fSpy to calculate the camera data(Fig. 1.6)

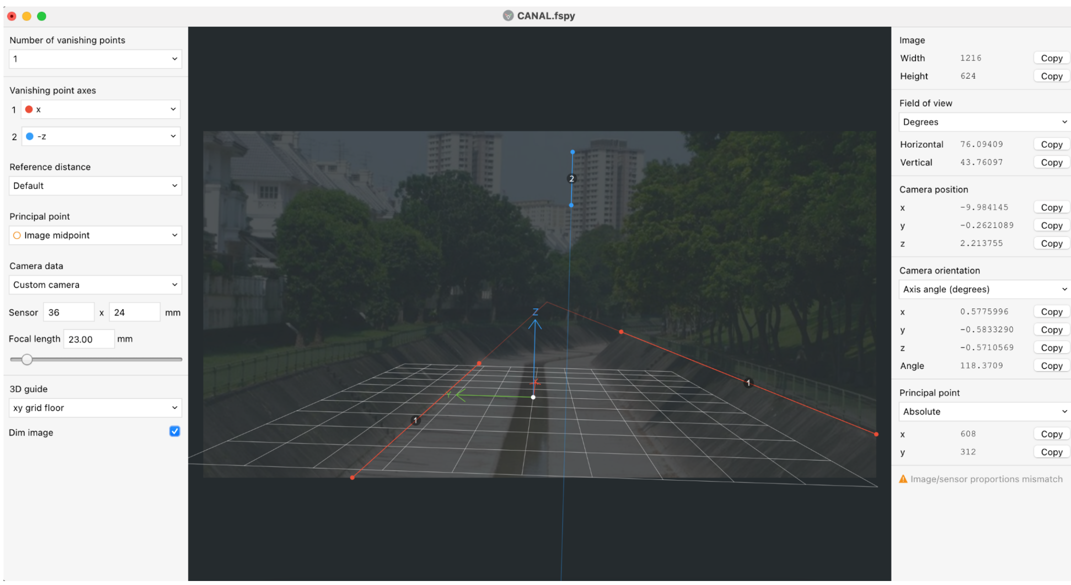

Fig. 1.6 - the fSpy software being used to calculate the camera data

After that, we can import the camera data into Blender and then model a trapezium prism to occupy the volume in the canal.

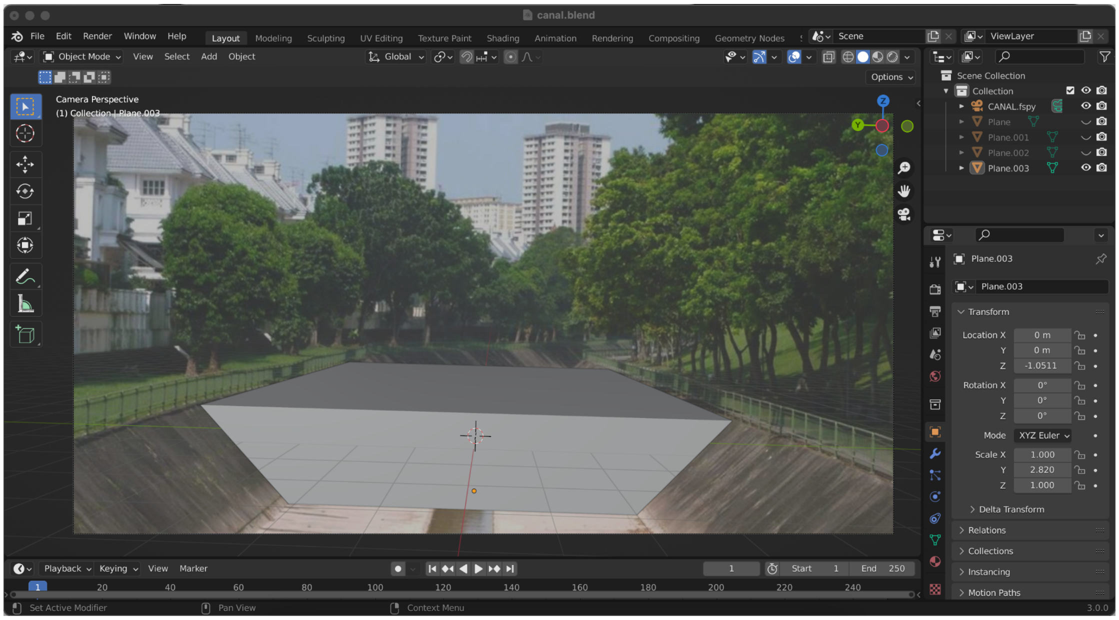

Fig. 1.7 - The trapezium prism inside the canal

However, this model is not to actual scale with the actual canal, but we can assume that the trapezium face of this prism is similar to the cross-section model that we have, which can allow us to get our measurements.

Blender comes inbuilt with a measuring tool(Fig. 1.8), and using the width we measured earlier, we can measure the width of this trapezium and calculate the scaling factor.

$$S = \frac{w}{9.15665}$$
$$= 2.38(3.s.f)$$

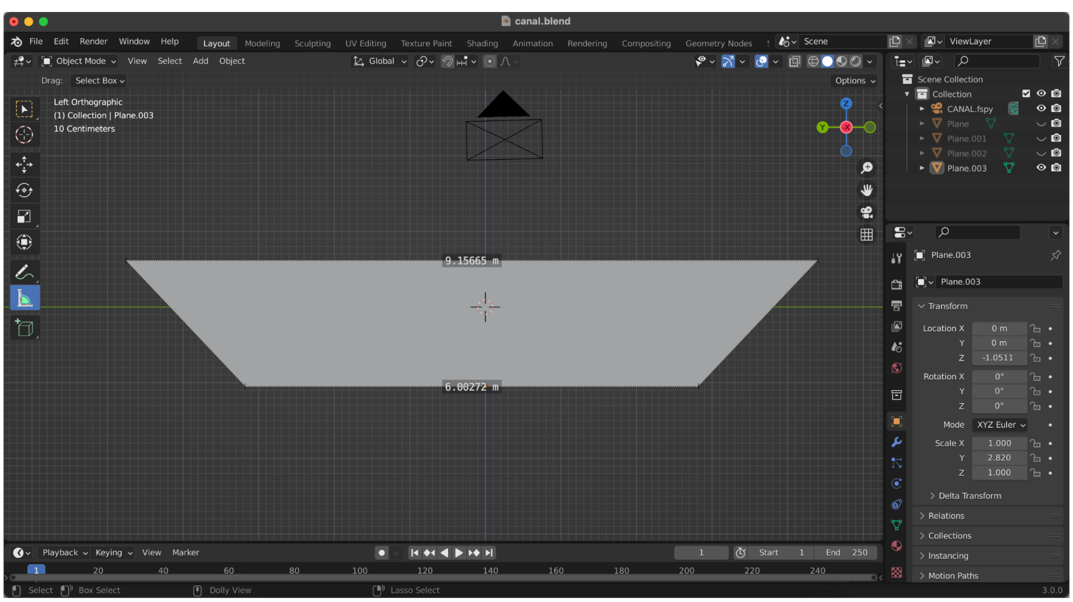

Fig. 1.8 - Blender's inbuilt measurement system

Thus, every side on the cross-section model is 2.38 times bigger than the 3D model. Using the scaling factor, we can find the width of the canal floor by multiplying it by the scaling factor.

$$6.00272S$$

We can find the run of the sloped lines by first taking the difference between the two parallel sides then dividing by 2 to get the final distance for the run.

$$w - 6.00272S$$

$$\frac{w-6.00272S}{2}$$

We can then divide this by the rise of the line, which is the depth of the canal, giving us

$$m = \frac{d}{\frac{w-6.00272S}{2}}$$

$$m = 1.33(3.s.f)$$

Now, we have all 3 of our variables with their correct values, all that's left is finding the area and perimeter of this cross-section model.

1.4 Calculating data from our model

To find the area, all we need to do is use the formula for find the area of a trapezium, which is

$h(\frac{a+b}{2})$, where a and b are the parallel sides of the trapezium, and h is the height

We know that the height of our trapezium is the canal's depth, the length of the parallel sides are the inequalities for the functions of the cross-section and the width of the canal. Thus, the area of the trapezium is

$$A = d(\frac{w+(\frac{-0.5mw+d}{-m} - \frac{-0.5mw+d}{m})}{2})$$

$$A = 90.4(3.s.f)$$

We can also find the perimeter of the trapezium, which is just the sum of all three sides. First, we can find the length of the sloped side by using Pythagorean's Theorem.

$$a^2 + b^2 = c^2$$
$$rise^2 + run^2 = c^2$$
$$d^2 + (\frac{w-6.00272}{2})^2 = c^2$$
$$c = \sqrt{d^2 + (\frac{w-6.00272}{2})^2}$$

After that, we can double that since both sloped sides are equal and add on the length of the canal floor.

$$P = 2(\sqrt{d^2 + (\frac{w-6.00272}{2})^2}) + (\frac{-0.5mw+d}{-m} - \frac{-0.5mw+d}{m})$$
$$P = 26.8(3.s.f)$$

Now, we have the area and the perimeter of our original canal. All that's left is redesigning the canal cross-section to be the most optimal.

2.1 Optimisation Definition

First, let us define the optimal shape of the canal. An optimal canal should use the same amount of material as the original, which means it should have the same perimeter as the original canal while holding the most water, which means it should have the most area.

Let us try and decide what shape our canal cross-section should look like first to be the most optimal. We can start with a shape that can hold the most area while having the least perimeter; a circle.

Circles are optimal for the most area for a set perimeter because every point on the circumference is equidistant from the center. Moving any points closer to the center would result in a lesser area, while moving any points further away would require more perimeter. Another way to prove this is by imagining any polygon and inflating it with air. The edges of the polygon will start to round out until the edges reach maximum curvature, which will make the shape turn into a circle. (Fig. 2.1)

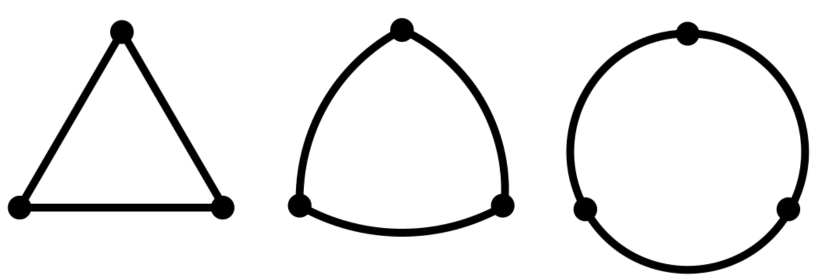

Fig. 2.1 - A diagram of a triangle's edges slowly increasing curvature until the shape is now a circle

However, from a practical point of view, circles may not be the best way for the cross-section of a canal since making curved surfaces are much more complex and are difficult for other modules to fit on, like the stairs in the canal photo.

Because of this, it might be better to use a regular polygon that is inscribed in a circle instead. To decide on which regular polygon to use, we can look back at the original shape of the canal.

As said earlier, the canal shape can be assumed to be a trapezium, but if we were to reflect the trapezium across the *x*-axis, we can see that it forms a hexagon. Thus, we can conclude that the canal shape should be half of a regular hexagon inscribed in a circle. Now, we just need to calculate the dimensions of the optimal canal.

2.2 Optimal Canal Dimensions›

Before we calculate the optimal canal, we need to figure out what variables we will need to get our optimised canal cross-section. Since we want our canal to be half a hexagon, we can find one side of the hexagon first.

Currently, we have variable P for the perimeter of our original canal. Because we know that the canal should be half a hexagon, and that we want to use the same perimeter for the optimal canal, we can divide P by 3, to get one side of the hexagon.

$$h = \frac{P}{3}$$

Let h be the side of the regular hexagon. We can reuse the previous functions for the original canal in the new optimal canal.

Starting with the canal floor, we know it should be a horizontal line that cuts the y-axis at the canal's depth, which is half the regular hexagon's height. To find this, we can use the property of regular hexagons that they are comprised of 6 equilateral triangles (Fig. 2.2)

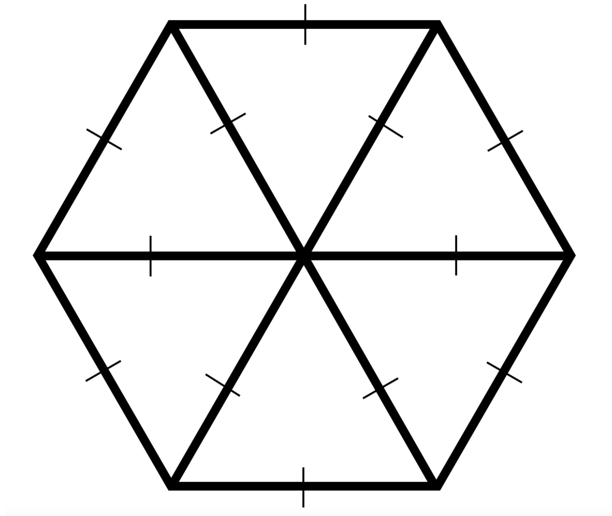

Fig. 2.2 - A regular hexagon divided into six congruent equilateral triangles

From Fig. 2.2, we can see that half the height of the regular hexagon is just the height of one of these equilateral triangles. Thus, if we focus on one of the triangles, we get Fig. 2.3

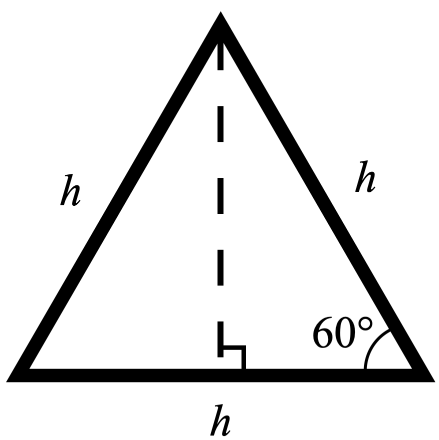

Fig. 2.3 - An equilateral triangle labeled

Now, all we need to do is use some trigonometry to help find the height of this triangle.

$$sin(a) = \frac{opp}{hyp}$$
$$sin(60) = \frac{height}{h}$$
$$height = h\,sin(60)$$

Thus, the height of the triangle is *h* sin(60°), which is the depth of our canal. Using the height of the triangle, we can form the function

$$y =- h\,sin(60)$$

for the floor of the canal. Next, we can form our absolute function for the sides of the canal. To do this, we first need to find the gradient of the sides. We already know the rise of the sides, which is the height of the canal. To find the run, we can use the same method from when we were trying to find the run of the original canal.

Thus,

$$y = |\frac{h\,sin(60)}{\frac{2h-h}{2}}x|$$
$$= |\frac{h\,sin(60)}{\frac{h}{2}}x|$$
$$= |\frac{h\,sin(60)}{0.5h}x|$$

We need to shift the function down to intersect the *x*-axis at half of the width like the original absolute function. Since we know that the width of the hexagon is $2h$, the function will intersect the *x*-axis at *h*. By using the same method as earlier, we can remove the absolute function and get

$$y = \frac{h\,sin(60)}{0.5h}x$$
$$y = \frac{h\,sin(60)}{0.5h}h$$
$$y = \frac{h\,sin(60)}{0.5h}h - \frac{h\,sin(60)}{0.5h}h$$

We can then simplify the function to get

$$y = \frac{h\,sin(60)}{0.5h}h - (\frac{h\,sin(60)}{0.5h} \times \frac{h}{1})$$
$$y = \frac{h\,sin(60)}{0.5h}h - (\frac{h \times h\,sin(60)}{0.5h})$$
$$y = \frac{h\,sin(60)}{0.5h}h - (\frac{h\,sin(60)}{0.5})$$
$$y = \frac{h\,sin(60)}{0.5h}h - (\frac{h\,sin(60)}{1} \div \frac{1}{2})$$
$$y = \frac{h\,sin(60)}{0.5h}h - (\frac{h\,sin(60)}{1} \times \frac{2}{1})$$
$$y = \frac{h\,sin(60)}{0.5h}h - 2h\,sin(60)$$
$$y = |\frac{h\,sin(60)}{0.5h}x| - 2h\,sin(60)$$

Now, we just need to implement the inequalities again. For the horizontal line, since we know that the length of it should be *h*, we can just have the inequality

$$y = -\ h\,sin(60)\{-\ 0.5h\ <\ x\ <\ 0.5h\}$$

For the absolute function, like the original function, we can use the canal's depth for the inequality.

$$y\ =\ |\frac{h\,sin(60)}{0.5h}x|\ -\ 2h\,sin(60)\{0\ >\ y\ > -\ h\,sin(60)\}$$

Now, when we plot these new functions, they look like this (Fig. 2.4)

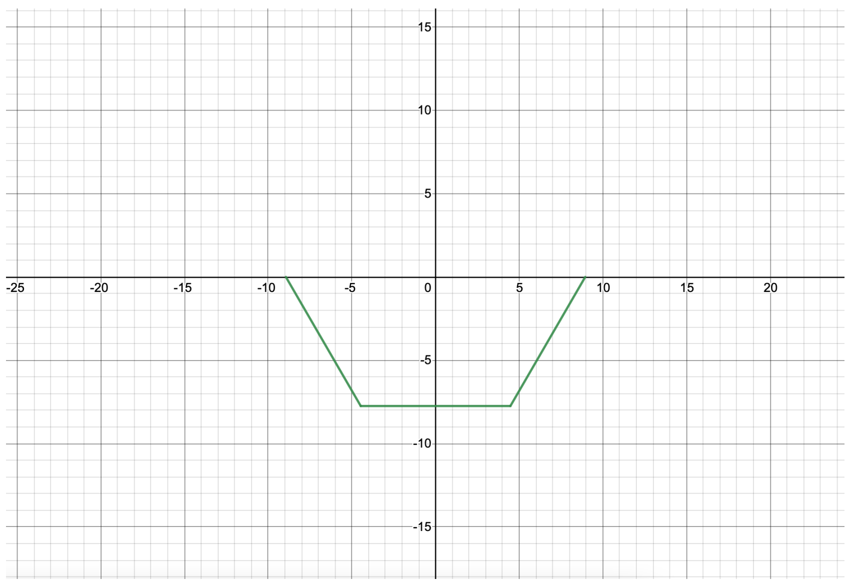

Fig. 2.4 - The new optimised canal cross-section

We now have the most optimal canal to hold the most water, but how much have we improved on the original canal design?

2.3 Canal Evaluation

Before evaluating our new canal design, let us calculate the area first.
Using the same formula as earlier, we can find out the area of our canal cross-section.

$$a\ =\ h(\frac{a+b}{2})$$
$$=\ h\,sin(60)(\frac{2h+h}{2})$$
$$=\ h\,sin(60)(\frac{3h}{2})$$
$$=\ h\,sin(60)(1.5h)$$
$$=\ 104\ (3.s.f)$$

Now, we can calculate the percentage increase of area to see how much we have improved on the original canal.

$$\frac{a-A}{A} \times 100\% = 14.9\% \text{ increase}$$

From the calculations, we can see that we have managed to hold up to 14.9% more water compared to the old canal. If we were to calculate the difference in the area, we get

$$a - A = 13.5m^2 (3.s.f)$$

Multiplying this by the length of the canal, which is estimated to be 7 kilometres[1], we get

$$13.5m^2 \times 7km = 13.5m^2 \times 7000m$$
$$= 94\,500m^3$$

Thus, our new canal design can hold up to 94 500 cubic meters more water.

2.4. Canal Practicality

We can now check how well our new canal design can manage rainfall for the next ten years.

From the NCCS (National Climate Change Secretariat) website, the annual rainfall has been increasing at an average rate of 67mm per decade[2] from 1980 to 2019. Assuming that the rainfall keeps increasing at a rate of 67mm linearly, using the MSS (Meteorological Service Singapore) 2019 Annual Climate Report[3]'s data of 1368mm of annual rainfall, we can use the data to form the function

$$y - y_1 = m(x - x_1)$$
$$y - rainfall = rate\ of\ rainfall\ increase\ per\ year(x - year)$$
$$y - 1368 = \frac{67}{10}(x - 2019)$$
$$y = 6.7(x - 2019) + 1368$$
$$y = 6.7x - 6.7(2019) + 1368$$
$$y = 6.7x - 13527.3 + 1368$$
$$y = 6.7x - 12159.3$$

Using this function, we can project the data to get an estimate on how much rainfall there would be per year. Substituting in 2023 for x, we get

$$y = 6.7x - 12159.3$$
$$y = 6.7(2032) - 12159.3$$
$$y = 1455.1$$

Thus, after 10 years, the annual rainfall will be around 1455.1mm, or 1.4551m. Since we know that the depth of our new canal is h sin(60), which is about 7.74m (3.s.f), it can hold the average rain for the next decade.

3.1 Conclusion

After all of the calculations, we managed to take the problem of flash floods and use coordinate geometry to find the original canal, redesign it to be the most optimal, then evaluate the effectiveness of the redesign by comparing it with the original canal and predicting rainfall data for the next 10 years. Overall, this method is quite adequate, but other methods like drainage tanks could reduce the risk of flash floods.

4.1 Resources

[1]https://walkingsingapore.com/ulu-pandan-park-connector-scenic-walk-ghim-moh-commonwealth-ave-west-boon-lay-way/
(We assume that the park connector is the same length as the canal, which is about 7 kilometers)
[2]https://www.nccs.gov.sg/singapores-climate-action/impact-of-climate-change-in-singapore/
[3]http://www.weather.gov.sg/wp-content/uploads/2020/03/Annual-Climate-Assessment-Report-2019.pdf
(only viewable on Google Chrome)

Final graph to scale: https://www.desmos.com/calculator/qi0ndiqldw
Final graph with sliders to change variables: https://www.desmos.com/calculator/pnq4ekkmyw